






# A Security Analysis of Restricted Syndrome Decoding Problems

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**Abstract.** Restricted syndrome decoding problems (R-SDP and R-SDP( $G$ )) provide an interesting basis for post-quantum cryptography. Indeed, they feature in CROSS, a submission in the ongoing process for standardizing post-quantum signatures. This work improves our understanding of the security of both problems. Firstly, we propose and implement a novel collision attack on R-SDP( $G$ ) that provides the best attack under realistic restrictions on memory. Secondly, we derive precise complexity estimates for algebraic attacks on R-SDP that are shown to be accurate by our experiments. We note that neither of these improvements threatens the updated parameters of CROSS.

**Keywords:** Code-based Cryptography · Restricted Errors · Post-Quantum Cryptography · Cryptanalysis

## 1 Introduction

It is well-known that large-scale quantum computers will be able to break most of the public-key cryptography in use today. The move to new post-quantum standards for signature and public-key encapsulation mechanisms (KEMs) is well underway. Indeed, the (U.S.) National Institute for Standards and Technology (NIST) has recently concluded a multi-year standardization process for post-quantum algorithms, based on feedback from international academia, industry, and governmental organizations, and the documentation for new standards is being finalized at the time of writing [AAC<sup>+</sup>22]. The majority of the selected algorithms are based on the computational hardness of problems related to structured lattices, and NIST is currently looking to diversify its portfolio by standardizing schemes based on different hardness assumptions. For KEMs, there are still several candidates from the aforementioned standardization process that are being evaluated, however, there were no remaining viable signature candidates. This prompted NIST to issue a call for additional post-quantum signature schemes, resulting in 40 proposed algorithms that were published in July 2023 for further scrutiny.

One of the main directions in post-quantum cryptography is to turn computationally hard problems from coding theory into digital signatures using zero-knowledge (ZK) proofs. For the resulting signature schemes, including CROSS, the public key is a description of an instance of the problem, and its (usually unique) solution is the secret key. The problem of choice in this case is typically a variant of the syndrome decoding problem (SDP), whose computational hardness is therefore at the core of the cryptographic security. The basic version of this problem is defined as follows.

**Problem 1** ((Computational) SDP). *For a given full-rank matrix  $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$ , a vector  $\mathbf{s} \in \mathbb{F}_q^{n-k}$  and an integer  $t \leq n$ , find  $\mathbf{e} \in \mathbb{F}_q^n$  of weight  $t$  satisfying  $\mathbf{e}\mathbf{H}^\top = \mathbf{s}$ .*

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State-of-the-art information set decoding (ISD) algorithms for solving this problem usually involve searching for a number of zero entries in  $\mathbf{e}$ . Thus their computational cost generally worsens as the weight is increased. This motivated the works of [BBC<sup>+</sup>20, BBP<sup>+</sup>24] to look into ways of relaxing the weight restriction by instead limiting the error vector  $\mathbf{e}$  to a subset of  $\mathbb{F}_q^n$ . The culmination of these works is CROSS [BBB<sup>+</sup>23], a family of signature schemes that was submitted to the ongoing call for additional post-quantum signature standards. The signature schemes are derived from an interactive zero-knowledge identification protocol using Fiat-Shamir transforms. The underlying hard problem in these protocols is either the *restricted syndrome decoding problem* (R-SDP) or a further specialization known as R-SDP( $G$ ). The interested reader can find a brief overview of the ZK protocol used in CROSS, and how the restricted decoding problems feature in it, in Appendix A.

The idea of R-SDP is to limit the entries of  $\mathbf{e}$  to a multiplicative subgroup  $E \subset \mathbb{F}_q^*$  of order  $z < q - 1$ . The restricted syndrome decoding problem with respect to the group  $E$  is then defined as

**Problem 2** (R-SDP). *Given  $g \in \mathbb{F}_q^*$  of order  $z$ ,  $E := \{g^j, j \in \{0..z-1\}\}$ , a full-rank matrix  $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$  and a vector  $\mathbf{s} \in \mathbb{F}_q^{n-k}$ , find  $\mathbf{e} \in E^n$  such that  $\mathbf{e}\mathbf{H}^\top = \mathbf{s}$ .*

The further specialization of R-SDP( $G$ ) is achieved by considering errors from a subgroup  $G$  of  $E^n$ . The set  $E^n$  is endowed with the  $\star$  operation, which performs the entry-wise multiplication of two vectors. We can then use elements  $\mathbf{a}_1, \dots, \mathbf{a}_m \in E^n$  to generate a subgroup  $(G, \star) \subset (E^n, \star)$  as

$$G := \langle \mathbf{a}_1, \dots, \mathbf{a}_m \rangle := \{\mathbf{a}_1^{u_1} \star \dots \star \mathbf{a}_m^{u_m} \mid u_i \in \{0, \dots, z-1\}\}.$$

The syndrome decoding problem restricted to  $G$  is now defined as follows.

**Problem 3** (R-SDP( $G$ )). *Given a subgroup  $(G, \star) \subset (E^n, \star)$  of order  $z^m$ , a full-rank matrix  $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$  and a vector  $\mathbf{s} \in \mathbb{F}_q^{n-k}$ , find  $\mathbf{e} \in G$  such that  $\mathbf{e}\mathbf{H}^\top = \mathbf{s}$ .*

The authors of [BBB<sup>+</sup>23] point to several advantages of using the restricted variants of SDP. Both R-SDP and R-SDP( $G$ ) are NP-hard, and generic decoders seem to have a larger computational complexity when compared to a similar instance of SDP. That said, the restricted variants are fairly recent problems, whose concrete security is not as well-studied as that of traditional syndrome decoding. One particular problem is how an attacker may use the structure of  $G$  to speed up variants of Stern-Dumer collision attacks on R-SDP( $G$ ). Another natural question is whether the restrictions on the error can open up for efficient algebraic attacks, which has recently been shown to be the case for a different decoding problem that relies on structured errors [BØ23].

**Contributions.** This work explores two directions in the security of restricted syndrome decoding problems. First, we present a new collision attack on R-SDP( $G$ ) that is designed with the special structure of  $G$  in mind. Note that the initial submission of CROSS (version 1.0) was not able to use  $G$  in their security analysis. This was, however, changed in an updated version 1.1, which was made public while we were working on this paper<sup>1</sup>. The new version proposes a collision attack that utilizes  $G$  and suggests updated parameters for R-SDP( $G$ ) to account for this. The collision attack presented in this paper is different from that of [BBB<sup>+</sup>23] and allows for different trade-offs in terms of time and memory complexity, and success probability.

The second half of this paper is devoted to algebraic attacks on R-SDP. Note that [BBB<sup>+</sup>23] provided a heuristic argument that Gröbner basis algorithms will not outperform

<sup>1</sup>As of July 2024, the latest release of the CROSS specification is version 1.2. However, the contents of this latter update has no impact on the topics in this paper.

other attacks, and provided experiments to support this claim. Our analysis goes beyond this by giving estimates for the Hilbert series of the equations of [BBB<sup>+</sup>23]. These estimates rely on standard arguments and have been shown to be exact in our experiments. The concrete bounds we can obtain from them suggest that the Gröbner basis cost conjectured in [BBB<sup>+</sup>23] might be slightly overestimated. Still, we agree with their overall conclusion that algebraic attacks will not threaten the CROSS R-SDP parameters.

## 2 A New Combinatorial Attack on R-SDP( $G$ )

In this section we introduce a new combinatorial attack on the R-SDP( $G$ ) problem, we report on a proof-of-concept implementation and we compare against the state of the art. We conclude that while our algorithm has a time complexity similar to that of state-of-the-art algorithms, it has much lower memory requirements and works for all keys, as opposed to some subset of weak keys.

**Exploiting group elements with disjoint support.** Recall that the R-SDP( $G$ ) problem is equivalent to finding a vector  $e$  in the intersection of the affine subspace defined by  $e\mathbf{H}^\top = \mathbf{s}$  and of the multiplicative group  $G$ . This appears to be a difficult problem because the additive structure of the affine subspace does not interact nicely with the multiplicative structure of  $G$ . However, observe that if  $\mathbf{l}, \mathbf{r} \in G$  are two elements of  $G$  with multiplicative support in the first and second half respectively, *i.e.*,  $\mathbf{l} := (\mathbf{u}, \mathbf{1}_{\lfloor n/2 \rfloor}) \in G$  and  $\mathbf{r} := (\mathbf{1}_{\lfloor n/2 \rfloor}, \mathbf{v}) \in G$  for  $\mathbf{u} \in E^{\lfloor n/2 \rfloor}$  and  $\mathbf{v} \in E^{\lfloor n/2 \rfloor}$ , then multiplication and addition does interact nicely. More precisely, we have

$$(\mathbf{l} \star \mathbf{r})\mathbf{H}^\top = (\mathbf{u}, \mathbf{v})\mathbf{H}^\top = \mathbf{u}\mathbf{H}_L^\top + \mathbf{v}\mathbf{H}_R^\top, \quad (1)$$

where  $\mathbf{H} = (\mathbf{H}_L \mathbf{H}_R)$ ,  $\mathbf{H}_L \in \mathbb{F}_q^{(n-k) \times \lfloor n/2 \rfloor}$ ,  $\mathbf{H}_R \in \mathbb{F}_q^{(n-k) \times \lfloor n/2 \rfloor}$ . We exploit this property to do a collision attack. Let  $L, R \subset G$  be the subgroups of  $G$  with support in the left and right half respectively. Then we try to find a collision

$$\mathbf{u}\mathbf{H}_L^\top = \mathbf{s} - \mathbf{v}\mathbf{H}_R^\top,$$

for  $(\mathbf{u}, \mathbf{v})$  such that  $\mathbf{l} = (\mathbf{u}, \mathbf{1}_{\lfloor n/2 \rfloor}) \in L$  and  $\mathbf{r} = (\mathbf{1}_{\lfloor n/2 \rfloor}, \mathbf{v}) \in R$ . If we find such a collision, then  $e := \mathbf{l} \star \mathbf{r}$  is a solution to the R-SDP( $G$ ) problem. The attack only works if there exists a solution in the subgroup  $L \star R := \{\mathbf{l} \star \mathbf{r} \mid \mathbf{l} \in L \text{ and } \mathbf{r} \in R\}$ , which in general only happens with a small probability. Therefore, we modify the attack to search for a solution in any coset  $L \star R \star \mathbf{f}$  and we run the attack for all  $\mathbf{f}$  in  $G/(L \star R)$  until a solution is found.

**Stern-Dumer-like optimization.** To reduce the cost of the attack, we use a standard idea inspired by the Stern-Dumer decoder [Dum91, Ste89]. More precisely, we put the matrix  $(\mathbf{H} \parallel \mathbf{s}^\top)$  in row-reduced echelon form and discard the  $\ell$  top rows. What remains is a new matrix  $\mathbf{H}' \in \mathbb{F}_q^{(n-k-\ell) \times n}$  and a new syndrome  $\mathbf{s}' \in \mathbb{F}_q^{n-k-\ell}$ . The idea is to sample solutions  $e'$  to the smaller system  $e'\mathbf{H}'^\top = \mathbf{s}'$  until we find a solution that also satisfies the original system  $e'\mathbf{H}^\top = \mathbf{s}$ . For appropriately chosen values of  $\ell$  this is more efficient than the direct approach because the cost of checking false positives is much smaller than the savings we get from attacking the smaller instance.

### 2.1 Description and Analysis of the Attack

A complete description of our attack can be found in Algorithm 1.

**Algorithm 1:** Algorithm for the R-SDP( $G$ ) problem

**Input** : Parity-check matrix  $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$ , syndrome  $\mathbf{s} \in \mathbb{F}_q^{n-k}$ , and subgroup  $G \subset E^n$  of rank  $m$ , parameter  $\ell$  such that  $n - \ell$  is even.

**Output** : Solution  $e \in G$  such that  $e\mathbf{H}^\top = \mathbf{s}$ , if it exists. Otherwise output  $\perp$ .

1 Using Gaussian Elimination mod  $z$ , compute the subgroups of  $G$ :

$$L := \{\mathbf{l} \in G, l_i = 1 \text{ for all } i \text{ such that } \ell + (n - \ell)/2 < i \leq n \},$$

$$R := \{\mathbf{r} \in G, r_i = 1 \text{ for all } i \text{ such that } \ell < i \leq \ell + (n - \ell)/2 \}.$$

2 Using elementary row operations, put the matrix  $(\mathbf{H} \parallel \mathbf{s}^\top)$  in the form

$$\begin{pmatrix} \mathbf{I}_\ell & * & * & * \\ \mathbf{0} & \mathbf{H}_L & \mathbf{H}_R & (\mathbf{s}')^\top \end{pmatrix},$$

where  $\mathbf{I}_\ell$  is the identity matrix of size  $\ell$ ,  $\mathbf{H}_L, \mathbf{H}_R \in \mathbb{F}_q^{(n-k-\ell) \times (n-\ell)/2}$ , and  $\mathbf{s}' \in \mathbb{F}_q^{n-k-\ell}$ .

3 for  $\mathbf{f} \in G/(L \star R)$  do

/\* Search for a solution in the coset  $L \star R \star \mathbf{f}$  \*/

4 Run a collision search to enumerate all pairs  $(\mathbf{l}, \mathbf{r}) \in L \times R$  such that

$$(\mathbf{f} \star \mathbf{l}) \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_L^\top \\ \mathbf{0} \end{pmatrix} = \mathbf{s}' - (\mathbf{f} \star \mathbf{r}) \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{H}_R^\top \end{pmatrix}.$$

5 For every collision  $(\mathbf{l}, \mathbf{r})$  check if  $(\mathbf{f} \star \mathbf{l} \star \mathbf{r})\mathbf{H}^\top = \mathbf{s}$ . If this is the case, output the solution  $e := \mathbf{f} \star \mathbf{l} \star \mathbf{r}$ .

6 Output  $\perp$ .

We give some more details on how to perform the steps of the algorithm below. A reader who is already comfortable with the algebraic and computational aspects of R-SDP( $G$ ) is invited to skip past these details.

**Step 1.** The first step computes the subgroups  $L$  and  $R$ , whose elements are the elements of  $G$  with ‘1’ entries at positions  $\ell + (n - \ell)/2 + 1$  to position  $n$ , and at positions  $\ell + 1$  to position  $\ell + (n - \ell)/2$  respectively. We assume (as in the CROSS specification) that the group  $G \subset E^n$  is given as input to the algorithm in the form of a full-rank matrix  $\mathbf{M} \in \mathbb{F}_z^{m \times n}$ , and a generator  $g$  of  $E$ , such that  $\mathbf{g} \in G$  if and only if  $\mathbf{g}$  is of the form  $g^{\mathbf{v}} = (g^{v_1}, \dots, g^{v_n})$  for some  $\mathbf{v} = (v_1, \dots, v_n)$  in the rowspan of  $\mathbf{M}$ . In other words, the multiplicative group  $G$  corresponds to an (additive) linear subspace of  $\mathbb{F}_z$  by taking component-wise discrete logarithms with respect to  $g$ , and the matrix  $\mathbf{M}$  is a generator matrix for this linear subspace.

The subgroup  $L$  corresponds to the subspace  $\mathcal{L} \subset \langle \mathbf{M} \rangle$  consisting of the vectors in  $\langle \mathbf{M} \rangle$  with zeros at position  $\ell + (n - \ell)/2 + 1$  up to position  $n$ . The space  $\mathcal{L}$  can be computed efficiently by doing Gaussian elimination on the matrix  $\mathbf{M}$ , choosing pivots from columns  $\ell + (n - \ell)/2 + 1$  to  $n$ , and then discarding the  $(n - \ell)/2$  rows from which the pivots are taken, since, by construction, they are the only rows that do not have the desired support. The remaining  $r = m + (\ell - n)/2$  rows are a basis for  $\mathcal{L}$ . A subspace  $\mathcal{R}$  can be constructed for  $R$  in a similar manner. Finally, we define the rank of  $G, L$ , and  $R$  as the dimensions of  $\langle \mathbf{M} \rangle, \mathcal{L}$ , and  $\mathcal{R}$  respectively.

**Step 2.** This step is just putting  $(\mathbf{H}||\mathbf{s}^\top)$  in row reduced echelon form using Gaussian elimination.

**Step 3.** To enumerate  $G/(L \star R)$ , let  $(\mathbf{b}_1, \dots, \mathbf{b}_r)$  and  $(\mathbf{b}_{r+1}, \dots, \mathbf{b}_{2r})$  be the bases for  $\mathcal{L}$  and  $\mathcal{R}$  computed in step 1. Then extend this to a basis  $\mathbf{b}_1, \dots, \mathbf{b}_m$  for  $\langle \mathbf{M} \rangle$  [Art10, Proposition 3.4.16 p. 89]. The elements of  $\langle \mathbf{b}_{2r+1}, \dots, \mathbf{b}_m \rangle$  enumerate uniquely the equivalence classes of  $\langle \mathbf{M} \rangle / (\mathcal{R} + \mathcal{L})$ , so one can efficiently enumerate all  $\mathbf{f} \in G/(L \star R)$  by iterating over  $\mathbf{f} = g^\mathbf{v}$  for all  $\mathbf{v} \in \langle \mathbf{b}_{2r+1}, \dots, \mathbf{b}_m \rangle$ .

**Step 4.** One can use any collision search algorithm to search for collisions between the left-hand side and right-hand side of the equation in line 4. We propose to use the van Oorschot-Wiener (vOW) algorithm [vW99], which is explained briefly in Appendix B.

**Correctness.** If there exists a solution  $\mathbf{e} \in G$  such that  $\mathbf{e}\mathbf{H}^\top = \mathbf{s}$ , then the algorithm is guaranteed to output a solution. Indeed, the for-loop on line 3 iterates over all elements of  $G/(L \star R)$ , which means that the algorithm will eventually reach an iteration with an  $\mathbf{f}$ -vector such that there exists a solution of the form  $\mathbf{e} := \mathbf{l} \star \mathbf{r} \star \mathbf{f}$ ,  $(\mathbf{l}, \mathbf{r}) \in L \times R$ . Since  $\mathbf{e}$  is a solution we have

$$(\mathbf{l} \star \mathbf{r} \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_L^\top \\ \mathbf{H}_R^\top \end{pmatrix} = \mathbf{s}',$$

which can be rewritten as

$$(\mathbf{l} \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_L^\top \\ \mathbf{0} \end{pmatrix} = \mathbf{s}' - (\mathbf{r} \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{H}_R^\top \end{pmatrix}. \quad (2)$$

This means that  $(\mathbf{l}, \mathbf{r})$  is one of the collisions found by the collision search on line 4. The solution  $\mathbf{e}$  will pass the check of line 5 and be output by the algorithm.

**Cost analysis.** We now analyze the expected cost of our attack. We assume that the  $\ell$  parameter is chosen such that  $n \leq 2m + \ell \leq 2n$ . Then, generically,  $L$  and  $R$  will have rank  $r = m + (\ell - n)/2$  and  $L \star R$  will have rank  $2r$ . We ignore the cost of computing these subgroups and doing the partial Gaussian elimination on  $(\mathbf{H}||\mathbf{s}^\top)$  because these operations can be done in polynomial time. The corresponding cost will in particular be negligible compared to the complexity of the rest of the attack for cryptographically interesting parameters.

The main cost of the algorithm is due to the loop on line 3. Heuristically, the solutions of the R-SDP( $G$ ) problem will be distributed uniformly over the cosets  $G/(L \star R)$ . If there are  $S$  such solutions, this means that on average we need to consider  $|G/(L \star R)|/(S+1) = z^{n-\ell-m}/(S+1)$  cosets before finding the first solution, since we are sampling cosets without replacement. The cost of each iteration is then the sum of the cost of collision search on line 4 and of the cost of checking for false positives on line 5.

**Cost of false positives.** Recall that  $\mathbf{e}'$  is said to be a false positive if it satisfies the smaller system  $\mathbf{e}'\mathbf{H}'^\top = \mathbf{s}'$ , with  $\mathbf{H}' \in \mathbb{F}_q^{(n-k-\ell) \times n}$ , but is not a valid solution to the decoding problem, i.e.,  $\mathbf{e}'\mathbf{H}^\top \neq \mathbf{s}$ . A random  $n$ -vector is expected to satisfy the smaller system with a probability of  $q^{k+\ell-n}$ , and hence we expect there to be, on average, approximately  $|L \star R|q^{k+\ell-n}$  false positives in each iteration of the for-loop in Algorithm 1. Dispelling a false positive  $\mathbf{e}'$  requires checking on average  $q/(q-1) \approx 1$  entries of  $\mathbf{e}'\mathbf{H}^\top = \mathbf{s}$ , which can be done with roughly  $n - \ell$  field multiplications. So the total cost of checking false positives in one iteration is approximately  $(n - \ell)z^{2r}q^{k+\ell-n}$  field multiplications.

**Cost of the collision search.** The cost of collision search depends on the strategy that is used to find the collisions. The naive method would be to build the list

$$\Gamma_{\mathbf{f}} := \left\{ (l, (l \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_L^\top \\ \mathbf{0} \end{pmatrix}) \text{ for } l \in L \right\},$$

and then for every  $\mathbf{r} \in R$  compute the right hand side of Equation (2) and check if it occurs in  $\Gamma_{\mathbf{f}}$ . An advantage of this approach is that the cost can be amortized over many  $\mathbf{f}$ . Indeed, after one  $\Gamma_{\mathbf{f}}$  is built, it will be cheaper to compute  $\Gamma_{\mathbf{f}'}$  for  $\mathbf{f}' := \mathbf{f} \star \mathbf{f}''$  where  $\mathbf{f}''$  is a low-weight codeword of  $G$ . Similarly, updating the set of right-hand sides is more efficient than recomputing it from scratch. However, because of the large memory requirement and the cost of accessing the huge list, we expect the naive approach to be much more expensive than more memory-friendly alternatives in “realistic” cost models. Therefore, we analyze the cost of our attack using the vOW collision search (see Appendix B) to enumerate the collisions.

Given two functions  $f_1 : X_1 \rightarrow Y$ ,  $f_2 : X_2 \rightarrow Y$ , with  $N = |X_1| = |X_2| < |Y|/2$  and enough memory to store  $M$  elements of  $X_1 \sqcup X_2$ , the vOW algorithm allows to enumerate a large fraction of all collisions  $(x_1, x_2) \in X_1 \times X_2$  such that  $f_1(x_1) = f_2(x_2)$  with a runtime cost dominated by  $3.5N^{1.5}/M^{0.5}$  evaluations of  $f_1$  and  $f_2$ . (Note: if  $M < N$ , then this compares favorably to the naive collision search which takes  $N^2/M$  function evaluations. Moreover, the vOW algorithm parallelizes much better than the naive approach.) Applying the vOW algorithm to our application, we have

$$\begin{aligned} N = |L| = |R| &= z^r = z^{m+(\ell-n)/2}, \\ f_1 : L &\rightarrow \mathbb{F}_q^{n-k-\ell} : l \mapsto (l \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_L^\top \\ \mathbf{0} \end{pmatrix}, \text{ and} \\ f_2 : R &\rightarrow \mathbb{F}_q^{n-k-\ell} : r \mapsto \mathbf{s}' - (r \star \mathbf{f}) \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{H}_R^\top \end{pmatrix}. \end{aligned}$$

An evaluation of  $f_1$  or  $f_2$  requires roughly  $(n-\ell)/2 \times (n-k-\ell)$  field multiplications, so the overall cost of the collision search can be estimated as the cost of  $3.5(n-\ell)(n-k-\ell)z^{1.5m+0.75(\ell-n)}/M^{0.5}$  field multiplications.

**Total cost.** Putting everything together, we estimate the total cost of the attack in field multiplications by

$$\underbrace{\frac{z^{n-\ell-m}}{S+1}}_{\text{iterations}} \left( \underbrace{3.5(n-\ell)(n-k-\ell) \frac{z^{1.5m+0.75(\ell-n)}}{M^{0.5}}}_{\text{cost of collision search}} + \underbrace{(n-\ell) \frac{z^{2r}}{q^{n-k-\ell}}}_{\text{cost of false positives}} \right). \quad (3)$$

## 2.2 Proof-of-Concept Implementation

We implemented our attack for the SL1 parameters of CROSS. Our analysis suggests that  $\ell = 15$  gives the best attack performance, resulting in groups  $L$  and  $R$  of rank 5, and  $\mathbf{H}_L$  and  $\mathbf{H}_R$  having 4 rows. Therefore, the naive collision search would require storing  $z^5$  elements of  $\log(p^4)$  bits each, which amounts to roughly 138 GB of memory. While this is not a prohibitively large amount of memory, we expect that the cost of frequently accessing such a large amount of memory makes the naive collision search less efficient

**Table 1:** Performance of our attack against the SL1 parameter set of CROSS, as a function of the amount of memory that is used.

Memory	fraction of distinguished points	$F$ evals per second	partial solutions per second
128 KB	$2^{-10}$	34 M	2.0 K
512 KB	$2^{-9}$	34 M	4.1 K
2 MB	$2^{-8}$	33 M	8.1 K
8 MB	$2^{-7}$	33 M	16 K
32 MB	$2^{-6}$	32 M	31 K
128 MB	$2^{-6}$	31 M	56 K
512 MB	$2^{-5}$	29 M	104 K
2 GB	$2^{-4}$	24 M	167 K
8 GB	$2^{-3}$	19 M	246 K

than the vOW method, so we used the latter in our implementation of the attack. The vOW method repeatedly evaluates the function  $F : \mathbb{F}_p^4 \rightarrow \mathbb{F}_p^4 : \mathbf{v} \mapsto f_{H(\mathbf{v})}(E_{H(\mathbf{v})}(\mathbf{v}))$ , where  $H : \mathbb{F}_p^4 \rightarrow \{1, 2\}$  is a hash function, and  $E_1 : \mathbb{F}_p^4 \rightarrow L$ , and  $E_2 : \mathbb{F}_p^4 \rightarrow R$  are arbitrary injective functions. The vOW algorithm repeatedly starts from random points and iterates this function until a distinguished point is reached, where in our implementation we say a point  $\mathbf{v} \in \mathbb{F}_p^4$  is distinguished if its bit-representation starts with a given number of zeros. Using a single core of an Intel i9-1088H CPU, our preliminary implementation can do up to roughly 34 million evaluations of  $F$  per second. Table 1 reports the amount of partial solutions that our implementation finds per evaluation of  $F$  (including the cost of checking if the partial solution is a solution to the full R-SDP( $G$ ) problem), as a function of how much memory the attack is allowed to use. The table shows that, as the amount of memory increases, fewer evaluations of  $F$  are required per partial solution. However, when the amount of memory is large, the number of  $F$ -evaluations per second decreases, because the attack is starting to get bottlenecked by the memory accesses, which get more expensive and more frequent (because of the higher fraction of distinguished points). The fraction of distinguished points is chosen to be the power of  $1/2$  that maximizes the number of partial solutions checked per second (last column). Since an expected number of  $q^\ell \approx 2^{134}$  partial solutions needs to be checked before a real solution is found, we do not claim that our attack breaks the security level of the new parameter set. E.g. using 8 GB of memory, the attack should take  $2^{134}/246000$  core-seconds, which is much longer than a key search against AES-128 would take on the same hardware.

### 2.3 Comparison with Previous Analysis

This section aims to discuss and compare our new collision attack with the recently introduced collision attack from [BBB<sup>+</sup>23, Section 7.1.2]. For completeness, we provide a brief overview of this latter attack in Appendix C. Our attack shares some similarities with the attack of [BBB<sup>+</sup>23], both algorithms use the Stern-Dumer-like approach in combination with a collision search. The main difference is that our algorithm performs collision searches between lots of functions with “small” domains, as opposed to the algorithm of [BBB<sup>+</sup>23] which does a single collision search between two functions with much larger domains. If the collision searches are performed naively, then both attacks have a similar time cost, but our attack has a much lower memory cost. In a realistic scenario where an attacker has a limited amount of memory it is necessary to perform the collision search with a time-memory trade-off such as with the vOW algorithm. In this case, for a fixed amount of



memory, our attack will be more time-efficient. A second difference is that our algorithm works for all instances of the RSDP- $(G)$  problem (i.e. all public keys of CROSS), whereas the algorithm of [BBB<sup>+</sup>23] only works for a small subset of weak public keys. This is because their algorithm searches for and exploits two large subcodes  $\mathcal{C}_1, \mathcal{C}_2$  of the dual of  $\log(G)$  which have disjoint support, and these codes do not exist for most  $G$ . We remark that it is possible to run the attack of [BBB<sup>+</sup>23] with  $\mathcal{C}_1 = \mathcal{C}_2 = \{0\}$ , which would result in an attack that works for all public keys, at the cost of only a relatively small loss of efficiency.

### 2.3.1 Comparing Complexity Estimates

In Table 2 and 3 we showcase for each CROSS parameter sets three instantiations of Algorithm 1, with different amounts of available memory. Let  $M$  be the number of distinguished points stored by the algorithm. In the first case we impose no restrictions on memory, and report  $M = N$ , where  $N$  is the size of the list when the algorithm parameters are chosen to optimize for time. In the two remaining instantiations we limit  $\log_2 M$  to  $\lambda/4$  and  $\lambda/8$ , where  $\lambda$  is the AES-security level associated with the parameter set. Time complexities are counted in  $\log_2$  bit operations, which is obtained by multiplying (3) with  $2 \log_2(q)^2$ , since we estimate the cost of one  $\mathbb{F}_q$ -multiplication as  $2 \log_2(q)^2$  bit operations, as is customary in the multivariate cryptography literature. We note that this is a pure bit-cost estimate that does not take into account the potential costs of accessing a (still fairly) large amount of memory, as discussed in Section 2.2. We have used  $S = z^m q^{k-n} + 1$  as the expected number of solutions for the R-SDP( $G$ ) problem.

Table 2 focuses on the updated parameter sets from version 1.1 of the CROSS specification. The different parameter sets are denoted  $SL(n, k, m)$ , where  $q = 509$  and  $z = 127$  are used in all sets. Recall that the security levels 1, 3 and 5 are based on the security of AES- $\lambda$  for  $\lambda = 128, 192$  and  $256$ , respectively. [BBB<sup>+</sup>23] estimates the AES variants to achieve the security of roughly 143, 207 and 271 bit operations, respectively.

**Table 2:** Comparison of complexity estimates for solving R-SDP( $G$ ) for the parameter sets in the CROSS submission. For each parameter set we compare the attack of [BBB<sup>+</sup>23] against our attack with 3 different bounds on the memory used by the vOW collision search. Time cost is given as the base-2 logarithm of the estimated number of bit operations.

Parameter Set	Attack	Time cost	Memory cost	$\log_2(\text{Succ. prob.})$
SL1(55, 36, 25)	[BBB <sup>+</sup> 23]	143	126	-118
	Alg.1 ( $\ell = 15$ )	152	35	0
	Alg.1 ( $\ell = 15$ )	154	32	0
	Alg.1 ( $\ell = 15$ )	162	16	0
SL3(79, 48, 40)	[BBB <sup>+</sup> 23]	210	187	-116
	Alg.1 ( $\ell = 23$ )	219	84	0
	Alg.1 ( $\ell = 23$ )	230	48	0
	Alg.1 ( $\ell = 25$ )	238	24	0
SL5(106, 69, 48)	[BBB <sup>+</sup> 23]	272	252	-209
	Alg.1 ( $\ell = 30$ )	283	70	0
	Alg.1 ( $\ell = 30$ )	284	64	0
	Alg.1 ( $\ell = 30$ )	300	32	0

For comparison, Table 2 also includes the time-optimized complexities reported in [BBB<sup>+</sup>23, Section 8.2]. The  $\log_2 M$  reported for this algorithm refers to the number of elements in the smaller of the two lists used in this attack. The success probability is given



**Table 3:** Same as Table 2, but for the CROSS parameters in the first version of the CROSS submission. All attacks succeed with probability 1.

Parameter Set	Attack	Time cost	Memory cost
SL1(42, 23, 24)	Alg.1 ( $\ell = 12$ )	121	63
	Alg.1 ( $\ell = 14$ )	134	32
	Alg.1 ( $\ell = 14$ )	140	16
SL3(63, 35, 36)	Alg.1 ( $\ell = 17$ )	177	91
	Alg.1 ( $\ell = 19$ )	195	48
	Alg.1 ( $\ell = 21$ )	203	24
SL5(87, 47, 48)	Alg.1 ( $\ell = 27$ )	231	126
	Alg.1 ( $\ell = 29$ )	255	64
	Alg.1 ( $\ell = 31$ )	268	32

by a conservative upper bound on the fraction of the public keys affected by this specific attack, computed as  $P(d_1, j_1)P(d_2, j_2)$  in accordance with [BBB<sup>+</sup>23, Section 7.1.2]. We note that compared to the attack of [BBB<sup>+</sup>23], our attack uses a dramatically reduced amount of memory and has a success probability of 1 rather than an extremely small success probability. This comes at the cost of a slightly increased time cost.

Table 3 shows the performance of Algorithm 1 against the older CROSS parameter sets from version 1.0 of the CROSS specification document. Memory restrictions have been chosen as for Table 2. Note that these parameters were shown to be vulnerable against the collision attack of [BBB<sup>+</sup>23] (version 1.1), though the authors did not suggest specific attack parameters for these cases; this is why we have chosen only to include the results from Algorithm 1 in Table 3.

**Discussion.** Note that the timings taken from [BBB<sup>+</sup>23] are computed under various conservative assumptions, that differ from the assumptions used in this paper. While memory does affect these time estimates to some extent, there are otherwise no restrictions on the amount of memory used in the attack (see [BBB<sup>+</sup>23, Theorem 15] for further details). For this reason, we emphasize that the numbers given for the different algorithms in Table 2 are meant to provide a qualitative – as opposed to a direct – comparison.

In Table 2 we see that the respective time complexities increase with Algorithm 1 as we add restrictions on memory. Thus the updated parameters seem to have a notable security buffer against these collision attacks under realistic memory restrictions. The older CROSS-parameters are, on the other hand, known to be insecure against the collision attack of [BBB<sup>+</sup>23] (version 1.1). Table 3 shows that these parameters are also vulnerable to Algorithm 1 under certain memory restrictions (and without assumptions on the public key).

### 3 Gröbner Basis Approach on R-SDP

This section is devoted to the study of algebraic attacks on R-SDP using Gröbner bases. We refer to [CLO13] for the fundamental definitions and theory on Gröbner bases, as well as their role in solving systems of multivariate polynomials. In the following we briefly recall the algebraic analysis from [BBB<sup>+</sup>23], which proposed the following way to model R-SDP as a system of polynomial equations.

**System 1** ([BBB<sup>+</sup>23], §7.2 p. 43). *Let  $R = \mathbb{F}_q[e_1, \dots, e_n]$ , where  $e_i$  is the  $i$ -th coordinate of the error vector  $\mathbf{e}$  for  $i \in \{1..n\}$ . Let  $\mathcal{L}$  be the linear system of size  $n - k$  corresponding*

to the parity-check equations  $e\mathbf{H}^\top = \mathbf{s}$  and let

$$\mathcal{Z} := \{\forall i \in \{1..n\}, e_i^z - 1\}.$$

Finally, let  $\mathcal{F} := \mathcal{L} \cup \mathcal{Z}$ .

It is easy to see that the solutions to  $\mathcal{F}$  exactly correspond to the solutions of the R-SDP problem. The analysis of [BBB<sup>+</sup>23] assumes that the complexity of finding a solution to  $\mathcal{F}$  is dominated by finding a Gröbner basis for the ideal  $\langle \mathcal{F} \rangle$ . The authors then estimate the complexity of solving this problem with the Gröbner basis algorithm  $F_5$  [Fau02] terminating in degree  $d_{\text{reg}}$  by

$$\mathcal{O}\left(\binom{n+d_{\text{reg}}}{n}^\omega\right), \quad (4)$$

where  $2 \leq \omega \leq 3$  is the linear algebra constant. Predicting the degree  $d_{\text{reg}}$  is, in general, difficult, but [BBB<sup>+</sup>23] argues heuristically that its growth will be linear in  $n$ . This is also backed by experiments performed with the computer algebra system Magma [BCP97], reported in [BBB<sup>+</sup>23, Table 2 p. 45]. In turn, the authors argue that such an algebraic approach could not be competitive with combinatorial methods for the proposed parameters in CROSS.

In this section we ultimately reach the same conclusion, but provide a tighter analysis. We also consider hybrid approaches, which seem to give better results in practice. We start with recalling some algebraic preliminaries, which will be needed for our analysis.

### 3.1 Preliminaries

Let  $R$  denote a polynomial ring over  $\mathbb{F}_q$  in  $n$  variables. For a set of polynomials  $p_1, \dots, p_m \in R$ , we let  $I = \langle p_1, \dots, p_m \rangle \subset R$  denote its ideal.  $I$  is said to be a homogeneous ideal if it can be generated by a set of homogeneous polynomials. Let  $R_d$  denote the  $\mathbb{F}_q$ -vector space generated by the monomials of degree  $d$  in  $R$ . For a homogeneous ideal  $I$ , we have the subspace  $I_d := \{p \in I, \deg(p) = d\} = I \cap R_d$ . The *Hilbert function* of  $R/I$  is then defined as

$$\begin{aligned} \mathcal{H}_{R/I} : \mathbb{N} &\longrightarrow \mathbb{N} \\ d &\longmapsto \dim_{\mathbb{F}_q}(R_d/I_d) \end{aligned}$$

and the *Hilbert series* is

**Definition 1** (Hilbert series). Let  $I \subset R$  be a homogeneous ideal. The Hilbert series of the quotient ring  $R/I$  is

$$\mathcal{H}_{R/I}(x) := \sum_{d=0}^{\infty} \mathcal{H}_{R/I}(d)x^d.$$

Computing the Hilbert series of a general ideal is difficult. There are, however, an important class of polynomial systems, known as *semi-regular sequences*, where an expression for the Hilbert Series is known. Recall that a homogeneous ideal  $I$  is said to be zero-dimensional if  $R/I$  is a finite dimensional vector space. In this case, the *degree of regularity*,  $d_{\text{reg}}$ , is the smallest integer  $d$  such that  $I_d = R_d$ .

**Definition 2** (Semi-regular sequence, [Bar04]). Let  $\mathcal{P} := \{p_1, \dots, p_m\}$  be a sequence of homogeneous polynomials such that  $I := \langle \mathcal{P} \rangle$  is zero-dimensional with degree of regularity  $d_{\text{reg}}$ . The sequence  $\mathcal{P}$  is said to be semi-regular if  $I \neq R$  and if for  $1 \leq i \leq m$ ,  $g_i p_i = 0$  in  $R/\langle p_1, \dots, p_{i-1} \rangle$  with  $\deg(g_i p_i) < d_{\text{reg}}$  implies  $g_i = 0$  in  $R/\langle p_1, \dots, p_{i-1} \rangle$ .

**Proposition 1** ([Bar04]). *Let  $\mathcal{P} := \{p_1, \dots, p_m\}$  be a homogeneous semi-regular system where  $\deg(p_i) = d_i$  for  $1 \leq i \leq m$  and let  $S_{m,n}(z) = \frac{\prod_{i=1}^m (1-x^{d_i})}{(1-x)^n}$ . Then the Hilbert Series associated with  $\mathcal{P}$  is given by*

$$\mathcal{H}_{R/\langle \mathcal{P} \rangle}(x) = [S_{m,n}(x)]^+,$$

where  $[\cdot]^+$  means truncation after the first non-positive coefficient.

The semi-regularity notion is extended to an affine sequence  $\{p_1, \dots, p_m\}$  by considering  $\{p_1^h, \dots, p_m^h\}$ , where  $p_i^h$  denotes the homogeneous part of  $p_i$  of maximal degree.

### 3.2 A Conjecture on the Hilbert Series

We consider the affine ideal  $\langle \mathcal{F} \rangle \subset R$  and  $I := \langle \mathcal{F}^h \rangle$  the homogeneous ideal generated by the highest degree parts. The degree of regularity of  $I$  will play a crucial role in our complexity estimates for finding solutions of  $\mathcal{F}$  (see Assumption 2 below). To derive this, we formulate a conjecture on the Hilbert series of  $R/I$ . Note that the ideal is guaranteed to be zero-dimensional due to the equations in  $\mathcal{Z}^h$ . In fact, we have a precise description of the Hilbert series of  $R/\langle \mathcal{Z}^h \rangle$ .

**Lemma 1.** *The Hilbert series of  $S := R/\langle \mathcal{Z}^h \rangle$  is equal to*

$$\mathcal{H}_S(x) = (1 + x + \dots + x^{z-1})^n = \left( \frac{1-x^z}{1-x} \right)^n.$$

*Proof.* As the equations from  $\mathcal{Z}^h$  are univariate and do not depend on the coordinate index, we obtain

$$\mathcal{H}_S(x) = (\mathcal{H}_{\mathbb{F}_q[e]/\langle e^z \rangle}(x))^n.$$

The fact that  $\mathcal{H}_{\mathbb{F}_q[e]/\langle e^z \rangle}(x) = 1 + x + \dots + x^{z-1}$  is clear.  $\square$

Our conjecture follows a similar strategy to [BØ23] and [CMT23, §5.3]. That is, we assume a generic behaviour of the  $\mathcal{L}^h$  equations in  $S = R/\langle \mathcal{Z}^h \rangle$ . More formally, we can define a regular property by adapting Definition 2 to this quotient.

**Definition 3** (Semi-regularity over  $S$ ). Let  $\mathcal{P} := \{p_1, \dots, p_m\}$  be a sequence of homogeneous polynomials in  $S$  such that  $I := \langle \mathcal{P} \rangle$  is zero-dimensional with degree of regularity  $d_{\text{reg}}$ . The sequence  $\mathcal{P}$  is said to be semi-regular if  $I \neq S$  and if for  $1 \leq i \leq m$ ,  $g_i p_i = 0$  in  $S/\langle p_1, \dots, p_{i-1} \rangle$  with  $\deg(g_i p_i) < d_{\text{reg}}$  implies  $g_i = 0$  in  $S/\langle p_1, \dots, p_{i-1} \rangle$ .

Note that while  $S$  is identical to the quotient defined by the top degree parts of the field equations from the field  $\mathbb{F}_z$ , the notion of semi-regularity over  $S$  is different from semi-regularity over  $\mathbb{F}_z$ . The main difference is that the coefficients of  $\mathcal{P}$  do not belong in  $\mathbb{F}_z$ . This means that we do not expect ‘‘Frobenius-like’’ cancellations caused by  $p^z$ . In particular, Definition 3 is different from semi-regularity over  $\mathbb{F}_2$  [Bar04, Definition 3.2.4] when  $z = 2$ .

**Assumption 1.** *We assume that the equations of the image of  $\mathcal{L}^h$  in  $S$  satisfy Definition 3.*

If this assumption holds, we obtain

**Proposition 2.** *Under Assumption 1, the Hilbert series of  $R/\langle \mathcal{F}^h \rangle$  is equal to*

$$\mathcal{H}_{R/\langle \mathcal{F}^h \rangle}(x) = \left[ (1-x)^{n-k} \left( \frac{1-x^z}{1-x} \right)^n \right]_+ = \left[ \frac{(1-x^z)^n}{(1-x)^k} \right]_+,$$

where  $[\cdot]_+$  refers to the truncation after the first non-positive coefficient.

*Proof.* The zero-dimensional character of the ideal  $\langle \mathcal{F}^h \rangle$  and Assumption 1 imply the relation

$$\mathcal{H}_{R/\langle \mathcal{F}^h \rangle}(x) = [(1-x)^{n-k} \mathcal{H}_S(x)]_+,$$

where  $[\cdot]_+$  is the truncation after the first non-positive coefficient. This follows from a reasoning similar to [Bar04, §3.3.1], or more recently [BØ23]. We conclude by Lemma 1.  $\square$

While we are not able to prove Assumption 1, it was found to be valid in our computations of Hilbert series for several parameter sets, see Appendix D. Thus we will, in practice, use Proposition 2 as a conjecture on the full Hilbert series  $\mathcal{H}_{R/\langle \mathcal{F}^h \rangle}$ .

### 3.3 A Tighter Complexity Bound

The degree of regularity and Equation (4) capture the complexity of computing Gröbner bases for *homogeneous* polynomial systems. In general, a more suited measure of the degree reached for *affine* polynomial systems in a Gröbner basis algorithm like  $F_4$  [Fau99] is the *solving degree* [CG23, Definition 1.1]. The exact relation between  $d_{\text{reg}}$  and the solving degree depends on the affine polynomials that are associated with non-trivial syzygies in  $\langle \mathcal{F}^h \rangle$ , and whether they reduce to zero modulo previous polynomials in the Gröbner basis computation. While there are polynomial systems where the solving degree and degree of regularity are different (see, e.g., [CG23, Example 4.3]), they typically coincide for polynomial systems that do not exhibit a particular algebraic structure. The two degrees are indeed found to be the same in all the experiments we have performed with the  $F_4$  algorithm implemented in Magma. This leads to the following assumption, which will justify our use of  $d_{\text{reg}}$  in complexity estimates.

**Assumption 2.** *We assume that the solving degree of  $\mathcal{F}$  coincides with the degree of regularity of the homogeneous ideal  $\langle \mathcal{F}^h \rangle$ .*

We have also verified that  $d_{\text{reg}}$  is equal to the first fall degree [CG23, Definition 1.3] for  $\mathcal{F}$  (which is consistent with Assumption 1).

Under Assumption 1 and 2, we now have an explicit way of computing the degree  $d_{\text{reg}}$  that is used in Equation (4). That is,  $d_{\text{reg}} = \deg(\mathcal{H}_{R/I}) + 1$ , where  $\mathcal{H}_{R/I}$  is the series in Proposition 2. However, we note that the binomial expression in Equation (4) counts all monomials of degree  $d_{\text{reg}}$  in  $n$  variables, which is an overestimate. Indeed, by first performing a reduction step modulo the  $e_i^z - 1 = 0$  equations, we may instead consider a matrix whose columns are indexed by monomials whose partial degree is only  $\leq z - 1$  in each variable, that is the monomials in  $S = R/\langle \mathcal{Z}^h \rangle$ . Recall that the coefficient of the degree  $d$  term in the series of Lemma 1 counts the number of degree  $d$  monomials in  $S$ . The number of degree  $\leq d$  monomials of this form is given by the coefficient of degree  $d$  in the following modification of this series.

$$\frac{1}{1-x} \mathcal{H}_S(x) = \frac{1}{1-x} \left( \frac{1-x^z}{1-x} \right)^n. \quad (5)$$

All in all, we can refine the cost estimate for computing a Gröbner basis of  $\mathcal{F}$  by

**Proposition 3.** *Under Assumption 1 and 2, we estimate the complexity of solving  $\mathcal{F}$  using Gröbner bases by*

$$\mathcal{O}(M_{z,(d_{n,k,z},n)}^\omega), \quad (6)$$

where  $d_{n,k,z}$  is the degree of regularity derived from the Hilbert series in Proposition 2,  $M_{z,(d_{n,k,z},n)}$  is the coefficient of degree  $d_{n,k,z}$  in the series of (5), and  $2 \leq \omega \leq 3$  is the linear algebra constant.

### 3.4 Hybrid Approach

In its plain form, the above attack performs poorly on the CROSS-R-SDP parameters of [BBB<sup>+</sup>23]. This motivates the study of hybrid techniques to improve the complexity. The standard hybrid approach corresponds to fixing several unknowns in  $\mathcal{F}$ . As the error vector is random in  $E^n$  and  $|E| = z$ , the success probability is  $1/z$  each time we fix a variable. Similarly to previous works, such as [Bet12, Section 4.2] and [BØ23], we adopt the same genericity assumptions as in the plain case regarding specialized systems. More specifically, we fix  $f$  of the  $k$  last variables, i.e.,  $e_i$ ,  $i \in \{n-k+1..n\}$ . Note that the top degree parts of  $\mathcal{P}$  will then be the same as the homogeneous polynomials associated with the parity-checks of the code defined by  $\mathbf{H}$  shortened at the same  $f$  positions. Since Assumption 1 was on the highest degree components, its hybrid adaptation given in Assumption 3 does not depend on the vector of specialization  $\mathbf{v} \in E^f$ , only on  $f$ . For any  $f \in \{0..k\}$ , let  $\mathbf{e}_f := (e_{f+1}, \dots, e_n)$ , let  $\mathcal{Z}_f := \{\forall i \in \{f+1..n\}, e_i^z - 1\}$  and let  $S_f := \mathbb{F}_q[\mathbf{e}_f]/\langle \mathcal{Z}_f^h \rangle$ .

**Assumption 3.** For any  $f \in \{0..k\}$  and any  $\mathbf{v} \in E^f$ , we assume that the system  $\mathcal{F}_{\text{spec},\mathbf{v},f}$  obtained by fixing the last  $f$  variables to  $\mathbf{v}$  is semi-regular in  $S_f$ .

In the same manner as Proposition 2, we can show the following result.

**Proposition 4.** Under Assumption 3, the Hilbert series of  $R/\langle \mathcal{F}_{\text{spec},\mathbf{v},f}^h \rangle$  is equal to

$$\mathcal{H}_{R/\langle \mathcal{F}_{\text{spec},\mathbf{v},f}^h \rangle}(x) = \left[ (1-x)^{n-k} \left( \frac{1-x^z}{1-x} \right)^{n-f} \right]_+ = \left[ \frac{(1-x^z)^{n-f}}{(1-x)^{k-f}} \right]_+,$$

where  $[\cdot]_+$  refers to the truncation after the first non-positive coefficient.

We also adopt an analogue to Assumption 2. Note that the following assumption does depend on the choice of  $\mathbf{v} \in E^f$ .

**Assumption 4.** We assume that the solving degree of  $\mathcal{F}_{\text{spec},\mathbf{v},f}$  coincides with the degree of regularity of the homogeneous ideal  $\langle \mathcal{F}_{\text{spec},\mathbf{v},f}^h \rangle$ .

Finally, we follow the reasoning of Section 3.3 and obtain

**Proposition 5.** Under Assumptions 3 and 4, we estimate the complexity of the standard hybrid approach on  $\mathcal{F}$  using Gröbner bases by

$$\mathcal{O} \left( \min_{\substack{0 \leq f \leq k \\ d_{n,k,z,f} \geq z}} \left( z^f M_{z,(d_{n,k,z,f},n-f)}^\omega \right) \right), \quad (7)$$

where  $d_{n,k,z,f}$  is the degree of regularity derived from the Hilbert series, where  $M_{z,(d_{n,k,z,f},n-f)}$  is the coefficient of degree  $d_{n,k,z,f}$  in the series  $\frac{1}{1-x} \left( \frac{1-x^z}{1-x} \right)^{n-f}$  and where  $2 \leq \omega \leq 3$  is the linear algebra constant.

On the CROSS parameters, this approach does not yield the most efficient attack. More precisely, for the 3 security levels, the best strategy using Equation (7) is to fix variables until we can solve at degree  $z = 7$ . This is the least degree where we can exploit the  $\mathcal{Z}$  equations. Concretely, as we fix almost all  $k$  variables, the cost of this approach is not competitive with the R-SDP adaptation of Prange's algorithm.

A possible generalization of this approach is to consider univariate equations of degree  $d_i \in \{1..z-1\}$  which vanish on the  $e_i$ 's with probability  $< 1$ . These equations correspond to guessing subsets  $E_i \subset E$  such that  $e_i \in E_i$ . (The method described above is then the case  $|E_i| = 1$ ). Note that a similar approach has already been performed on MQ systems, see [Bet12, §4.4 p. 111]. However, testing different values of  $d_i$  suggests that this generalization would still not outperform Prange on the CROSS parameters.

### 3.5 Asymptotic analysis

The goal of this section is to give an asymptotic equivalent of the degree of regularity when the length  $n$  tends to infinity assuming a constant code rate  $R := 1/\alpha := k/n$ . An initial observation is that the conjectured Hilbert series for  $\mathcal{F}$  is the same as the one of a semi-regular system containing  $n$  equations of degree  $z$  in  $k$  variables. Thus we can leverage the technical machinery from the work of Bardet, Faugère and Salvy [Bar04, BFS05].

**Theorem 1** ([Bar04], Theorem 4.1.3 p. 81). *For any constant  $\alpha > 1$ , an asymptotic equivalent of the degree of regularity of a semi-regular sequence of  $n = \alpha k$  equations with degrees  $d_1, \dots, d_n$  in  $k$  variables when  $n \rightarrow +\infty$  is*

$$d_{reg} \sim \phi(x_0)k,$$

where

$$\phi(x) = \frac{x}{1-x} - \frac{1}{k} \sum_{j=1}^n \frac{d_j x^{d_j}}{1-x^{d_j}}, \quad (8)$$

and where  $x_0$  is the root of  $\phi'$  such that  $\phi(x_0) > 0$  is minimal.

The function  $\phi$  relevant to the setting  $d_i = z$  for all  $i \in \{1..n\}$  is  $\phi(x) = \frac{x}{1-x} - \alpha \frac{z x^z}{1-x^z}$ . The consequence for the R-SDP setting of CROSS is

**Lemma 2.** *Assuming a constant code rate  $R := 1/\alpha$ , there exists a constant  $c_{\alpha,z} > 0$  such that the degree of regularity of  $\langle \mathcal{F}^h \rangle$  behaves as*

$$d_{n,k,z} \sim c_{\alpha,z}k.$$

We can precisely give the value of  $c_{\alpha,z}$  when  $z$  is small. We showcase this for the cases  $z = 2$  and 3. For the former case, we may rely on the study of quadratic equations performed in [Bar04].

**Lemma 3** ([Bar04], Corollary 4.4.1 p. 95). *When  $z = 2$ , an equivalent of the degree of regularity  $d_{reg}$  when  $k$  goes to infinity is  $d_{reg} \sim c_{\alpha,2}k$ , where*

$$c_{\alpha,2} = -\frac{1}{2} + \alpha - \sqrt{\alpha(\alpha-1)}.$$

The case  $z = 3$  can be obtained by a resultant computation, namely

**Lemma 4.** *When  $z = 3$ , an equivalent of the degree of regularity  $d_{reg}$  when  $k$  goes to infinity is  $d_{reg} \sim c_{\alpha,3}k$ , where*

$$c_{\alpha,3} = -\frac{1}{2} + \frac{3\alpha}{2} - \frac{\sqrt{81\alpha^2 - 24\sqrt{\alpha} - 54\alpha - 3}}{6}.$$

*Proof.* We have to study the roots of the derivative  $\phi'$ , where

$$\phi(x) = \frac{x}{1-x} - 3\alpha \frac{x^3}{1-x^3}.$$

More precisely, we look for the smallest possible value of  $u = \phi(x_0)$  for such a root  $x_0$ . Since  $\phi$  and  $\phi'$  are rational fractions, we may rewrite

$$\phi(x_0) - u = 0, \quad \phi'(x_0) = 0,$$

as a polynomial system

$$P(x_0, u) = 0, \quad Q(x_0, u) = 0,$$

where the polynomials  $P$  and  $Q$  can be easily computed. To eliminate  $x_0$ , we consider the resultant  $T(Y) = \text{Res}_X(P(X, Y), Q(X, Y))$  which is a polynomial of degree 4. Its roots sorted in decreasing order are

$$\left\{ \begin{array}{l} -\frac{1}{2} + \frac{3\alpha}{2} + \frac{\sqrt{81\alpha^2 + 24\sqrt{\alpha} - 54\alpha - 3}}{6} \\ -\frac{1}{2} + \frac{3\alpha}{2} + \frac{\sqrt{81\alpha^2 - 24\sqrt{\alpha} - 54\alpha - 3}}{6} \\ -\frac{1}{2} + \frac{3\alpha}{2} - \frac{\sqrt{81\alpha^2 - 24\sqrt{\alpha} - 54\alpha - 3}}{6} \\ -\frac{1}{2} + \frac{3\alpha}{2} - \frac{\sqrt{81\alpha^2 + 24\sqrt{\alpha} - 54\alpha - 3}}{6} \end{array} \right.$$

The two greatest roots are positive as  $\alpha > 1 > 1/3$ . The explicit expression of the resultant also shows that the product of roots is negative. This implies that we eventually keep the value

$$-\frac{1}{2} + \frac{3\alpha}{2} - \frac{\sqrt{81\alpha^2 - 24\sqrt{\alpha} - 54\alpha - 3}}{6}.$$

□

Degrees  $z \geq 4$  can still be tackled in the same fashion. The technical difficulty is that we will handle a resultant of larger degree and thus we are no longer guaranteed to have a closed form expression for its roots.

Finally, let us return to the hybrid approach. We note that the conjectured Hilbert series for  $R/\langle \mathcal{F}_{\text{spec}, v, f}^h \rangle$  is also the Hilbert series of a semi-regular system, which means that we can obtain a value for  $d_{\text{reg}}$  in the same manner as in the plain case. Based on this equivalence, a possible next step could be to derive the best asymptotic trade-off following the approach of [BFP09, Bet12] (see also [BS24, §4.4] for a recent use of this technique).

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## A The CROSS Signature Schemes

The CROSS signature schemes are based on a Fiat-Shamir transformation of a zero-knowledge (ZK) identification protocol. In the following we provide a brief description of these constructions, and how the restricted syndrome decoding problems feature in them. We refer to [BBB<sup>+</sup>23] for further details on implementation and improvement.

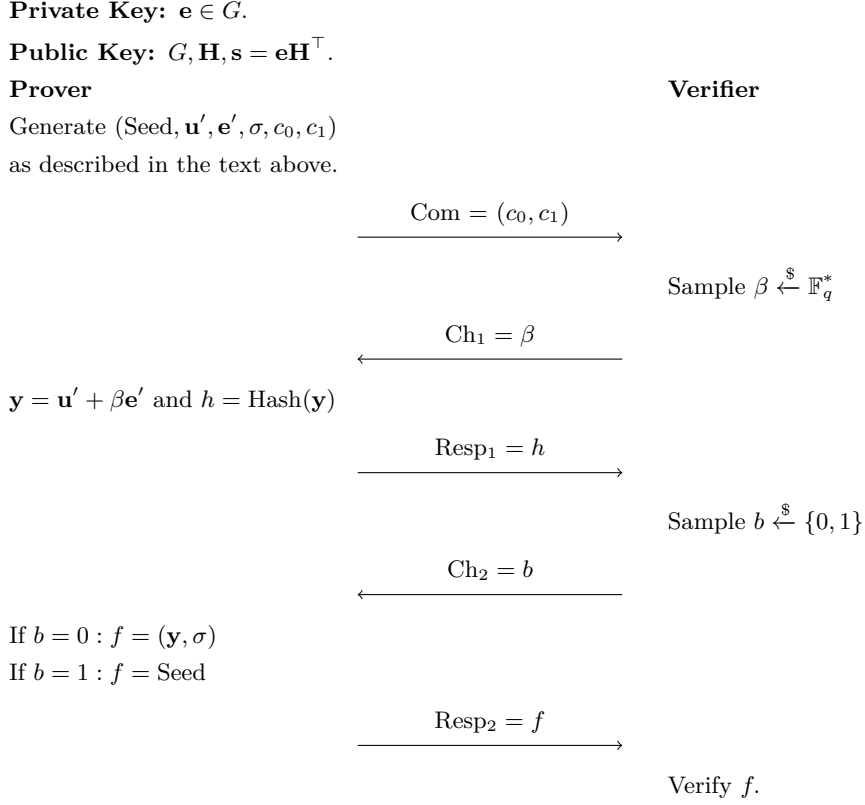
The ZK identification protocol at the core of the CROSS signature schemes, CROSS-ID (Figure 1), is a 5-pass protocol consisting of an initial commitment, and two challenge-response cycles. We give the following description for R-SDP( $G$ ) (Problem 3) as the underlying problem, noting that the case of R-SDP (Problem 2) follows by choosing  $G = E^n$ .

**Choice of Hard Problem.** The CROSS-ID protocol is an adaptation of a ZK-protocol originally intended for the SDP problem [CVE10]. [BBP<sup>+</sup>24] shows that signature sizes can be significantly decreased when instead using R-SDP and R-SDP( $G$ ), which has motivated the CROSS design.

**Setup.** The prover generates a subgroup  $(G, \star) \subset (E^n, \star)$  (where we recall that  $\star$  denotes the component-wise multiplication of vectors), an element  $\mathbf{e} \in G$  and a parity-check matrix  $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$ . The elements  $G, \mathbf{H}$  and  $\mathbf{s} = \mathbf{e}\mathbf{H}^\top$  are made public, while  $\mathbf{e}$  is kept secret. For a fixed hash function  $\text{Hash}(\cdot)$ , the prover further generates the following values.

1. Sample  $\text{Seed} \xleftarrow{\$} \{0, 1\}^\lambda$ , and  $\{\mathbf{e}', \mathbf{u}'\} \xleftarrow{\$} G \times \mathbb{F}_q^n$ ,
2. Find  $\sigma \in G$  such that  $\sigma \star \mathbf{e}' = \mathbf{e}$ ,
3. Compute  $\mathbf{u} = \sigma \star \mathbf{u}'$ , and  $\tilde{\mathbf{s}} = \mathbf{u}\mathbf{H}^\top$ ,
4. Compute  $c_0 = \text{Hash}(\tilde{\mathbf{s}}, \sigma)$ , and  $c_1 = \text{Hash}(\mathbf{u}', \mathbf{e}')$ .

The CROSS-ID protocol now proceeds as depicted in Figure 1.



**Figure 1:** Cross-ID.

**Verification.** At the end of CROSS-ID, the verifier proceeds in two possible ways, depending on  $b$ . If  $b = 0$ , then  $f = (\mathbf{y}, \sigma)$ . Note that

$$\sigma \star \mathbf{y}\mathbf{H}^\top = (\sigma \star \mathbf{u}' + (\beta\sigma) \star \mathbf{e}') \mathbf{H}^\top = \tilde{\mathbf{s}} + \beta\mathbf{s}.$$

Thus the verifier needs to check the equalities  $c_0 = \text{Hash}(\sigma \star \mathbf{y}\mathbf{H}^\top - \beta\mathbf{s}, \sigma)$ ,  $h = \text{Hash}(\mathbf{y})$ , and that  $\sigma$  is indeed an element of  $G$ .

If  $b = 1$ , then the verifier is given the seed, and is able to compute  $\mathbf{e}'$ ,  $\mathbf{u}'$  and  $\mathbf{y}$ . It now remains to verify the equalities  $c_1 = \text{Hash}(\mathbf{u}', \mathbf{e}')$  and  $h = \text{Hash}(\mathbf{y})$ .

**Signature Scheme.** The Fiat-Shamir transform [FS86] is a common technique for turning an interactive identity protocol, such as CROSS-ID, into a non-interactive signature scheme. The core idea is that the prover simulates the verifier by generating the challenges as the output of a hash function depending on the message and information used earlier in the protocol. The resulting value  $f$  acts as the signature of the message, which can be verified as outlined above.

In practice, this non-interactive process must be repeated a number of times to prevent forgery attacks. We refer to [BBB<sup>+</sup>23] for further details on parameter choices and security notions.

## B The van Oorschot-Wiener Algorithm

This appendix provides a brief overview of the main idea behind the van Oorschot-Wiener (vOW) algorithm for finding collisions. For further details and run-time analysis we refer to [vW99].

Given a suitable function  $F : Y \rightarrow Y$ , the goal of a collision search is to find two different values of  $Y$  that produce the same output under  $F$ . We start by fixing a defining property for a subset of elements in  $Y$ , which we will refer to as *distinguished* points. This distinguishing property should be easy to test, such as a fixed number of leading zeroes in the bit representation of an element.

**Trail Generation.** The collision search proceeds by constructing trails of elements in  $Y$ . A starting point  $x_0 \in Y$  is chosen, and a trail of points  $x_1, x_2, \dots$  is created by

$$x_i = F(x_{i-1}), \text{ for } i = 1, 2, \dots$$

This process is continued until it reaches a distinguished point  $x_d$ . The trail is stored in a list by its start and end point  $(x_0, x_d)$ .

**Collisions.** The collision search algorithm continues to add trails to the list, until two trails with the same distinguishing point,  $(x_0, x_d)$  and  $(x'_0, x_d)$  are found. At this point we are (hopefully) in the case depicted in Figure 2. The attacker can now recompute the two trails starting from  $x_0$  and  $x'_0$  until the collision  $F(x_i) = F(x'_j), x_i \neq x'_j$  is found. Note that there is a possibility that  $x'_0$  is in the trail  $(x_0, x_d)$ , which would not lead to a true collision. In this case,  $(x'_0, x_d)$  is discarded and the search continues.

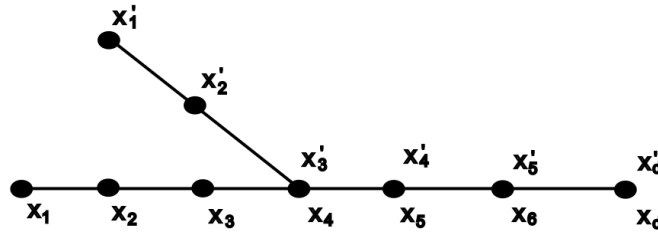


Figure 2: A collision in the vOW algorithm.

**Application to CROSS.** In the collision attack on CROSS described in Section 2, we do not have a single function  $F : Y \rightarrow Y$  with the same domain and range, but we rather have two functions  $f_1 : X_1 \rightarrow Y$  and  $f_2 : X_2 \rightarrow Y$ . The vOW algorithm can also be used in this situation via a reduction to the single-function case. One defines a new function  $F : Y \rightarrow Y$  as

$$F(y) = \begin{cases} f_1(E_1(y)) & \text{if } H(y) = 0 \\ f_2(E_2(y)) & \text{if } H(y) = 1 \end{cases},$$

where  $H$  is a hash function that outputs a single bit, and  $E_1, E_2$  are arbitrary injective encoding functions that maps elements of  $Y$  to  $X_1$  and  $X_2$  respectively. Then one uses the single-function vOW algorithm to find collisions for  $F$ . Heuristically, a collision  $F(y) = F(y')$  has a 50% chance that  $H(y) \neq H(y')$ , which means that the collision corresponds to a collision between  $f_1$  and  $f_2$ .

## C An Overview of the Collision Attack in [BBB<sup>+</sup>23]

We start with a brief, high-level overview of the collision attack introduced in version 1.1 of [BBB<sup>+</sup>23]. Consider the  $(n - m) \times n$  parity-check matrix  $\mathbf{M}_H$  of  $\mathbf{M}_G$ , and let  $\langle \mathbf{M}_H \rangle$  denote its associated code. For simplicity, let us suppose that there are two subcodes  $\mathcal{C}_1, \mathcal{C}_2 \subset \langle \mathbf{M}_H \rangle$  of dimensions  $d_1, d_2$  and disjoint support  $J_1$  and  $J_2$ , respectively. Moreover, we write  $j_1 := |J_1|$ ,  $j_2 := |J_2|$ , and assume for simplicity that the subcodes are chosen such that  $\rho := j_1 - d_1 = j_2 - d_2$ . We further introduce the notation

$$l := j_1 + j_2 - k, \quad \tilde{l} := 2\rho - m.$$

After a reordering of the columns, the subcode  $\mathcal{C}_1$  is generated by the matrix  $\begin{bmatrix} \mathbf{0} & \mathbf{G}_1 & \mathbf{0} \end{bmatrix}$ , for a matrix  $\mathbf{G}_1 \in \mathbb{F}_z^{d_1 \times j_1}$  representing the support  $J_1$ . Similarly, we have that  $\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{G}_2 \end{bmatrix}$  generates  $\mathcal{C}_2$  for some  $\mathbf{G}_2 \in \mathbb{F}_z^{d_2 \times j_2}$ . Upon performing suitable linear operations on the matrices  $\mathbf{M}_H$  and  $\begin{bmatrix} \mathbf{H} & \mathbf{s}^\top \end{bmatrix}$  one obtains

$$\mathbf{M}'_H = \begin{pmatrix} * & * & * \\ \mathbf{0} & \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2 \end{pmatrix}, \quad \mathbf{H}' = \begin{pmatrix} * & * & * & * \\ \mathbf{0} & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{s}'^\top \end{pmatrix} \quad (9)$$

for matrices  $\mathbf{M}_1 \in \mathbb{F}_z^{\tilde{l} \times j_1}$ ,  $\mathbf{M}_2 \in \mathbb{F}_z^{\tilde{l} \times j_2}$ ,  $\mathbf{H}_1 \in \mathbb{F}_q^{l \times j_1}$  and  $\mathbf{H}_2 \in \mathbb{F}_q^{l \times j_2}$ . The idea is now to make lists from the kernel elements of  $\mathbf{B}_1^\top$  and  $\mathbf{B}_2^\top$ , and use information from both  $\mathbf{M}'_H$  and  $\mathbf{H}'$  to search for collisions. More precisely, the two lists are created as

$$\left\{ (\mathbf{x}_1, \mathbf{x}_1 \mathbf{M}_1^\top, g^{(\mathbf{0} \mathbf{x}_1 \mathbf{0})} (\mathbf{0} \ \mathbf{H}_1 \ \mathbf{0})^\top) \mid \text{for } \mathbf{x}_1 \in \text{Ker}(\mathbf{B}_1^\top) \right\}, \quad (10)$$

$$\left\{ (\mathbf{x}_2, \mathbf{x}_2 \mathbf{M}_2^\top, \mathbf{s}'_2 - g^{(\mathbf{0} \mathbf{0} \mathbf{x}_2)} (\mathbf{0} \ \mathbf{0} \ \mathbf{H}_2)^\top) \mid \text{for } \mathbf{x}_2 \in \text{Ker}(\mathbf{B}_2^\top) \right\}.$$

The latter  $\mathbb{F}_z^{\tilde{l}} \times \mathbb{F}_q^l$  part of a list element is called the *label*, and is used to find collisions. Indeed, it can be verified that a solution to the R-SDP( $G$ ) problem will correspond to a  $\mathbb{F}_z$ -tuple whose last  $j_1 + j_2$  entries form an element  $(\mathbf{x}_1, \mathbf{x}_2) \in \text{Ker}(\mathbf{B}_1^\top) \times \text{Ker}(\mathbf{B}_2^\top)$ , where

$$\mathbf{x}_1 \mathbf{M}_1^\top + \mathbf{x}_2 \mathbf{M}_2^\top = \mathbf{0}; \text{ and}$$

$$g^{(\mathbf{0} \mathbf{x}_1 \mathbf{x}_2)} (\mathbf{0} \ \mathbf{H}_1 \ \mathbf{H}_2)^\top = \mathbf{s}'_2.$$

## D Extra Details on the Algebraic Attack

We ran several experiments in the computer algebra system Magma to verify Assumption 1 and 2. The results are reported in the following.

**F<sub>4</sub> algorithm.** In Tables 4 and 5, we compare the solving degree  $d_{\text{solv}}$  of  $\mathcal{F}$  to the first degree fall  $d_{\text{ff}}$ . Both quantities have been obtained from Magma's implementation of F<sub>4</sub>. We also indicate the *degree falls* generated at the step in degree  $d_{\text{ff}}$ . We notice that both degrees coincide in all cases. The computations for the larger instances were fairly time consuming. For instance, the total time to obtain the last rows in Tables 4 and 5 was 441852.309s and 227555.690s respectively.

**Hilbert series.** For the same parameters, we computed the Hilbert series of  $R/\langle \mathcal{F}^h \rangle$ . This series was always in accordance with the one of Proposition 2. The degree of regularity derived from it was also identical to the first degree fall  $d_{\text{ff}}$  from above.

**Table 5:** Parameters  $q = 127$ ,  $z = 7$  and  $k = n/2$ .

$n$	$d_{\text{ff}}$	$d_{\text{solv}}$
8	9 (6:15 7:40 8:5)	9
10	10 (7:54 8:145 9:15)	10
12	11 (8:245 9:490 10:45)	11
14	12 (9:1204 10:1631 11:77)	12
16	12 (9:2660 10:6896 11:3492)	12
18	13 (12:2925 13:12630)	13

$n$	$d_{\text{ff}}$	$d_{\text{solv}}$
8	13 (10:21 11:84 12:7)	13
10	14 (13:195 14:169)	14
12	16 (14:1833 15:456)	16
14	17 (16:3521 17:4816)	17